4.1 – Polynomials, Extrema & Symmetry

I.

POLYNOMIAL FUNCTION

A **polynomial function** P is given by

a_0x^0

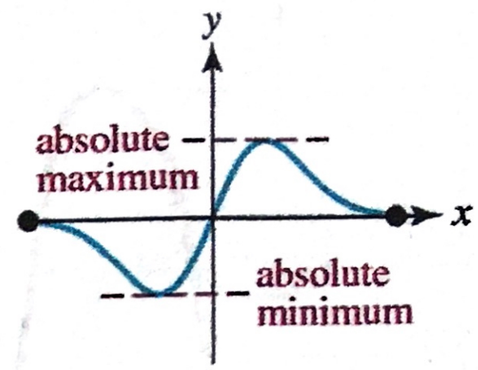
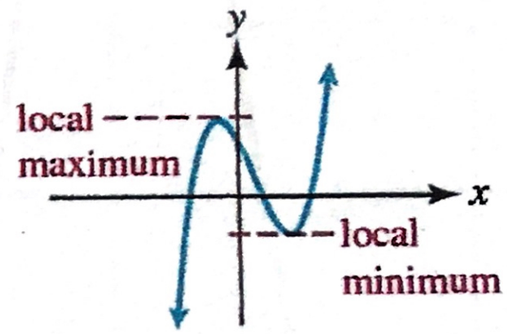
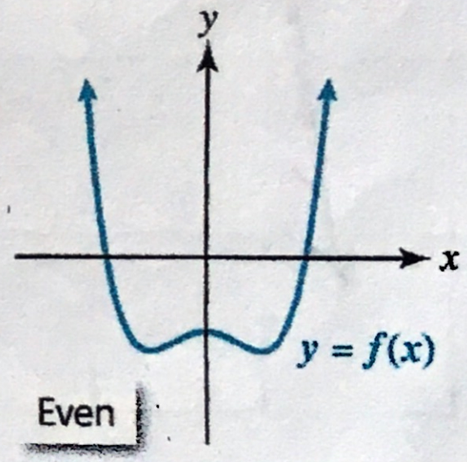
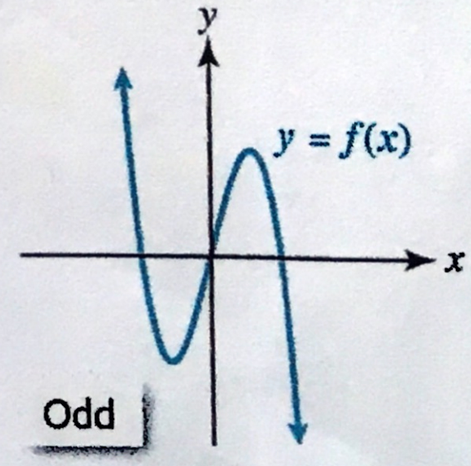
$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0,$$

where the coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and the exponents are whole numbers.

The "first" nonzero coefficient, a_n, is called the **leading coefficient.** The term a_nx^n is called the **leading term.** The **degree** of the polynomial function is n. Some examples of polynomial functions follow.

EXAMPLE	DEGREE	LEADING TERM	LEADING COEFFICIENT
$f(x) = 3 \quad (f(x) = 3 = 3x^0)$	0	3 or $3x^0$	3
$f(x) = \frac{2}{3}x + 5 \quad \left(f(x) = \frac{2}{3}x + 5 = \frac{2}{3}x^1 + 5\right)$	1	$\frac{2}{3}x$	$\frac{2}{3}$
$f(x) = 4x^2 - x + 3$	2	$4x^2$	4
$f(x) = x^3 + 2x^2 + x - 5$	3	x^3	1
$f(x) = -x^4 - 1.1x^3 + 0.3x^2 - 2.8x - 1.7$	4	$-x^4$	-1

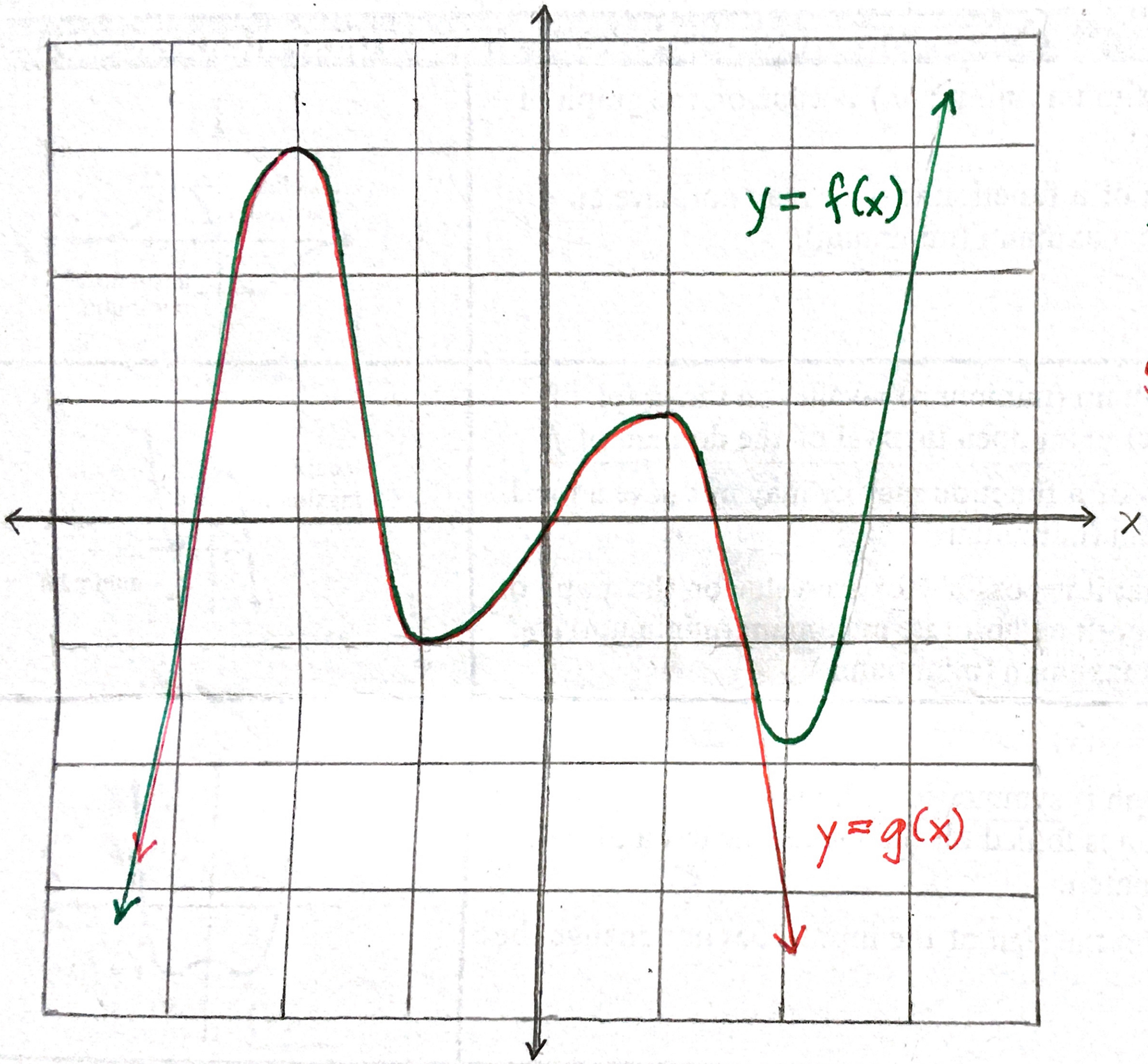
4.1 Putting It All Together

CONCEPT	EXPLANATION	GRAPHICAL EXAMPLE
Absolute, or global, maximum (minimum)	The maximum (minimum) y-value on the graph of $y = f(x)$ A graph of a function may or may not have an absolute maximum (minimum).	
Local, or relative, maximum (minimum)	A maximum (minimum) y-value on the graph of $y = f(x)$ in an open interval of the domain of f A graph of a function may or may not have a local maximum (minimum). Note that it is possible for a y-value on the graph of f to be both an absolute maximum (minimum) *and* a local maximum (minimum).	
Even function	$f(-x) = f(x)$ The graph is symmetric with respect to the y-axis. If the graph is folded on the y-axis; the left and right halves match. Changing the sign of the input does not change the output.	
Odd function	$f(-x) = -f(x)$ The graph is symmetric with respect to the origin. If the graph is spun about the origin, the graph reappears after half a turn, or 180°. Changing the sign of the input only changes the sign of the output.	

II. EXTREMA

III. SYMMETRY

II. IDENTIFY ALL EXTREMA.



	LOCAL		ABSOLUTE	
	MAX	MIN	MAX	MIN
f				
g				

III. (cont'd) EVEN, ODD or NEITHER ?

Symbolic (Algebraic) Test

EVEN: $f(-x) = f(x)$

ODD: $f(-x) = -f(x)$

(59) $f(x) = 5x$

(61) $f(x) = x + 3$

(65) $f(x) = x^4 - 6x^2 + 2$

(67) $f(x) = x^3 - 2x$

(69) $f(x) = x^2 - x^3$

(75) $f(x) = \dfrac{1}{1+x^2}$

(•) $f(x) = \sqrt{x}$, $f(x) = |x|$

4.2 – Turning-Point/Leading-Term Tests, Regression (cont'd)

I. TURNING-POINT TEST

Definition. A **turning point*** of a polynomial P is a point where P changes from increasing to decreasing (or vice versa).

1. If degree $(P) = n$, then: ← given formula
   (a) max (# TPs) $= n-1$
   (b) max (# x-intercepts) $= n$.
2. If #TPs $= n$, then min [degree (P)] $= n+1$. ← given graph

---

* Occurs at a local extremum.

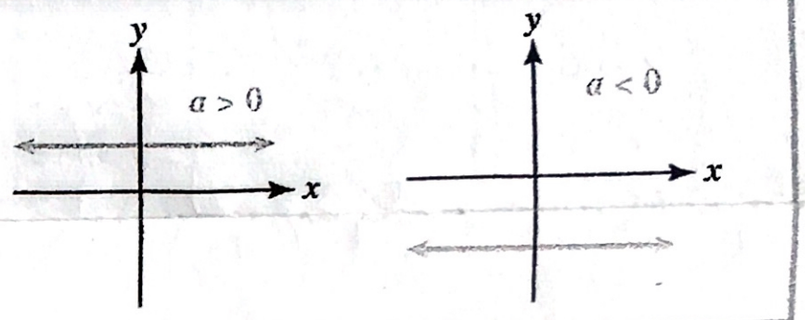
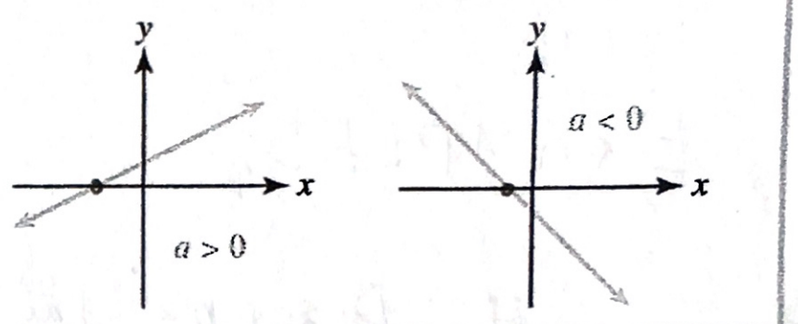
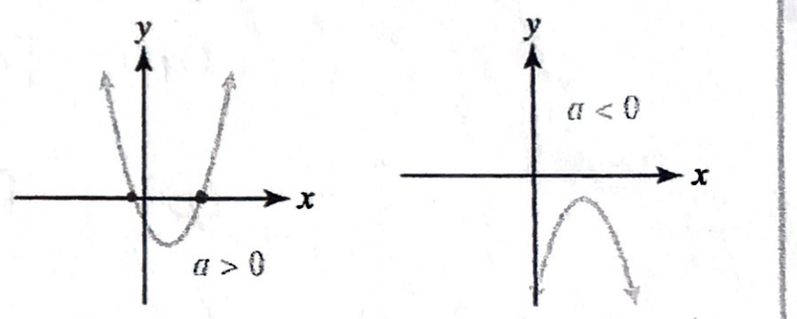
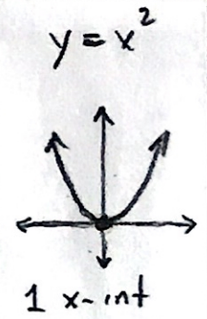
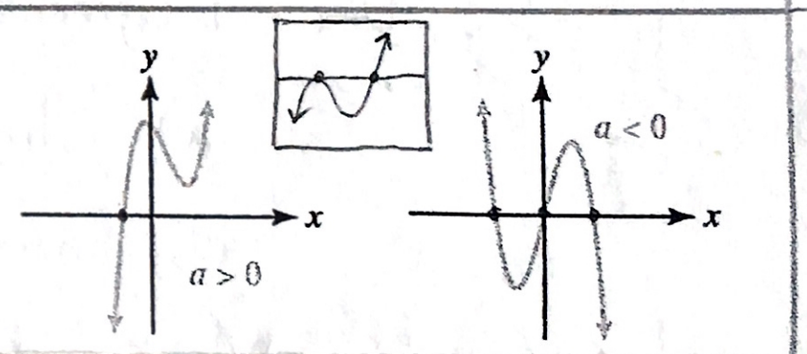
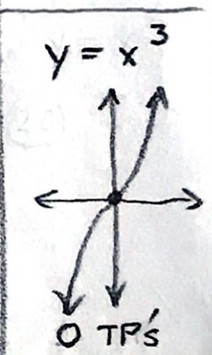
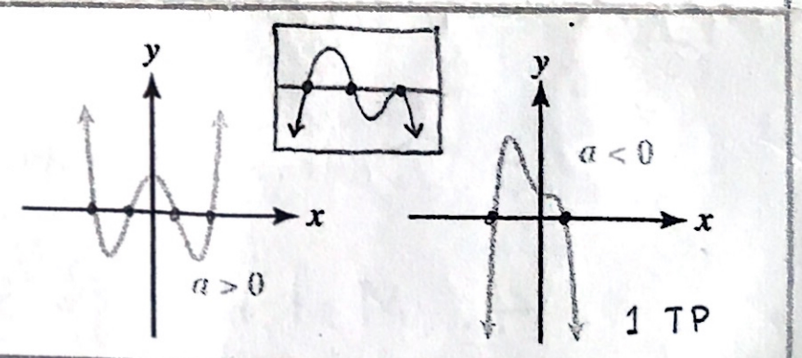
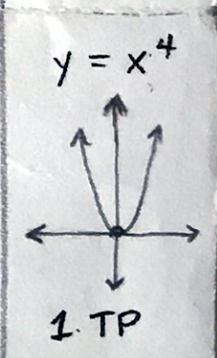
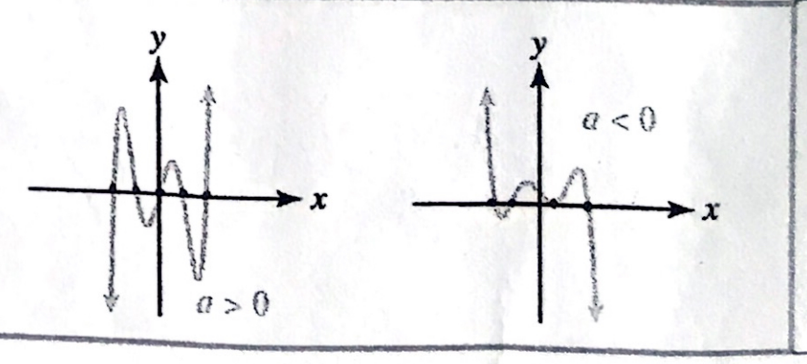
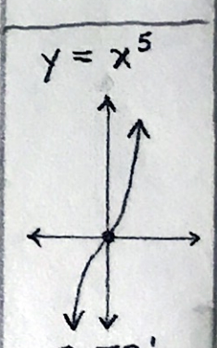
I. TURNING POINTS (cont'd)

TP – Point where f changes from ↗ to ↘ (or vice versa).

4.2 Putting It All Together

x-intercepts vs. turning points

Higher degree polynomials generally have more complicated graphs. Each additional degree allows the graph to have one more possible turning point and x-intercept. The graph of a polynomial function is continuous and smooth; it has no breaks or sharp edges. Its domain includes all real numbers. The end behavior of a polynomial always tends to either ∞ or $-\infty$. End behavior describes what happens to the y-values as $|x|$ becomes large.

FUNCTION TYPE	CHARACTERISTICS	EXAMPLE GRAPHS	
Constant (degree 0)	No x-intercepts and no turning points $f(x) = a, a \neq 0$		
Linear (degree 1)	One x-intercept and no turning points $f(x) = ax + b, a \neq 0$		BASE
Quadratic (degree 2)	At most two x-intercepts and exactly one turning point $f(x) = ax^2 + bx + c, a \neq 0$		
Cubic (degree 3)	At most three x-intercepts and up to two turning points $f(x) = ax^3 + bx^2 + cx + d, a \neq 0$		
Quartic (degree 4)	At most four x-intercepts and up to three turning points $f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$		
Quintic (degree 5)	At most five x-intercepts and up to four turning points $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + k, a \neq 0$		
* nth degree	At most ☐ x-intercepts and up to ☐ turning points.		

CONCLUSION

II. THE LEADING-TERM TEST (for describing "end" behavior)

If $a_n x^n$ is the leading term of a polynomial function, then the behavior of the graph as $x \to \infty$ or as $x \to -\infty$ can be described in one of the four following ways.

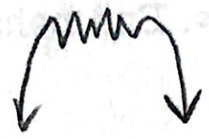
n	$a_n > 0$	$a_n < 0$
Even		
Odd		

The ∧∧∧∧ portion of the graph is not determined by this test.

II. Leading-Term Test (for describing "end" behavior; ie as $x \to \pm\infty$)

$(ax^n)^*$

* n = largest degree of polynomial

n	a > 0		a < 0	
even		as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$, $f(x) \to \infty$		as $x \to -\infty$, $f(x) \to -\infty$ as $x \to \infty$, $f(x) \to -\infty$
odd		as $x \to -\infty$, $f(x) \to -\infty$ as $x \to \infty$, $f(x) \to \infty$		as $x \to -\infty$, $f(x) \to \infty$ as $x \to \infty$, $f(x) \to -\infty$

EXAMPLES.

1. Describe the end behavior (graphically & symbolically) of:

   (a) $f(x) = 7x^5 - 2x^4 + x^2 - 5$

   (b) $f(x) = -x^8 + 2x + 1$

2. Use f (green) & g (red) functions to determine:

   (a) whether $a > 0$ or $a < 0$

   (b) number of T.P.s

   (c) minimum degree

	f (green)	g (red)
(a)	$a > 0$	$a < 0$
(b)	4	3
(c)	5	4

3. Determine max number of TPs in #1.

4. Match each of the following functions.

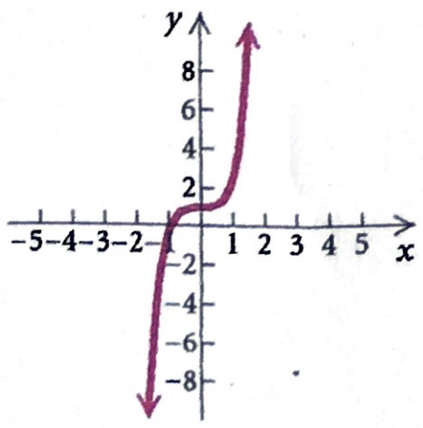
a) $f(x) = 3x^4 - 2x^3 + 3$

b) $f(x) = -5x^3 - x^2 + 4x + 2$

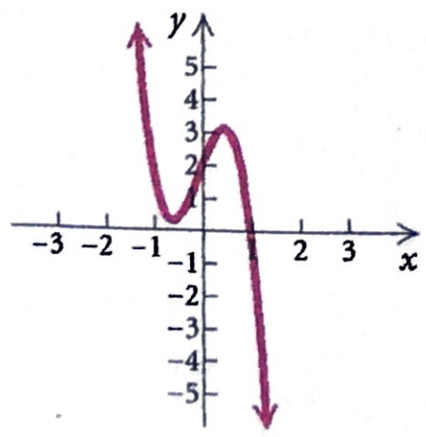
c) $f(x) = x^5 + \frac{1}{4}x + 1$

d) $f(x) = -x^6 + x^5 - 4x^3$

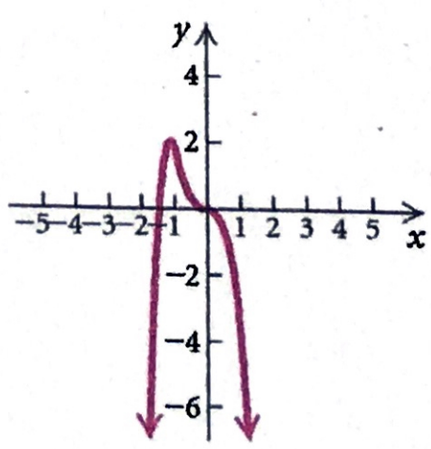
A.



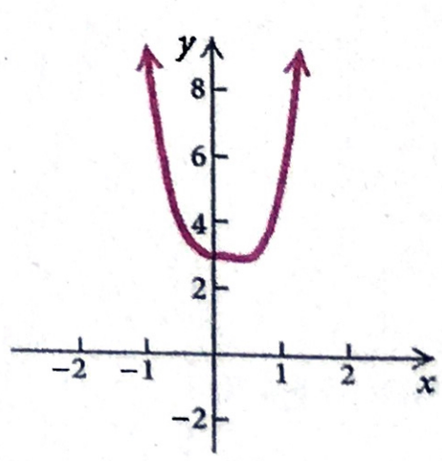
B.



C.



D.



III. 4.2 (cont'd) – Applications

CUBIC REGRESSION

105. Aging in America The table lists the number N (in thousands) of Americans over 100 years old for selected years x.

x	1960	1970	1980	1990	2000	2010
N	3	5	15	37	50	70

(a) Use regression to find a polynomial of degree 3 that models the data. Let $x = 0$ correspond to 1960. (round to six decimal places)

(b) Graph f and the data... together on paper.

(c)* Estimate N in 1994 and in 2020. ($x = 34, 60$)

(d) Did your estimates in part (c) involve interpolation or extrapolation?

(b) Use window:

-1	0
6	80
1	10

Answers. (a) $f(x) \approx -0.000630x^3 + 0.065437x^2 - 0.367593x + 2.817460$; (c) 41.2, 80.3; (d) interpolation, extrapolation

* Use TABLE feature, if possible.